5. THE COMPANION TO THE COMPANION. Since the equation (1) is of the form $Fy=\Psi(x)$, it is evidently included in the corresponding exponential equation

$$(Fy)^{1/Fy}=(\Psi x)^{1/\Psi x}. \tag{8}$$

The latter yields the equation of the first companion, and also

$$y^{1/y}=v^{1/(v-1)} \text{ and } (fx)^{1/fx}=v^{v/(v-1)}, \tag{9}$$

which are the equations of the companion to the companion.

We propose now to prove that this companion to the companion is in general an imaginary curve.

The minimum value of $v^{1/(v-1)}+v^{v/(v-1)}$ may be shown, by differentiation, to correspond to $v=1$; and therefore by equation (7) it is $2e$. Hence, for any value of v, one or the other of the expressions above is equal to or greater than e. Let it be the first one. Then

$$y^{1/y}\gtreqless e. \quad \therefore \frac{\log y}{y}\gtreqless 1. \tag{10}$$

But the maximum value of $(\log y)/y$ is readily shown to be $1/e$; and therefore the maximum value of $y^{1/y}$ is $e^{1/e}$. Equations (10) can therefore be satisfied only for imaginary values. If we suppose $v^{v/(v-1)}\gtreqless e$, we should, in the same way, be led to the introduction of imaginaries. Hence,

Companion curves, whose parent equations are expressed by a relation between two variables, themselves have no real companions.

Hereafter, by *companion*, we shall mean the real companion.

6. FORM OF THE EQUATION OF THE PARENT CURVE. The form and even the nature of the companion is often changed by a change in the *form* of the equation of the parent curve. In this way, by a change of origin and axes, an indefinite number of companions may be formed for the same parent curve. A simple illustration will be found by plotting the companions to $y=x$ and $y=a$, a constant.

7. TABLE OF THE v-EXPRESSIONS.

v	$v^{1/(v-1)}$	$v^{v/(v-1)}$	v	$v^{1/(v-1)}$	$v^{v/(v-1)}$
.1	12.910	1.291	2.1	1.963	4.132
.2	7.475	1.495	2.2	1.929	4.244
.3	5.576	1.673	2.3	1.898	4.378
.4	4.605	1.842	2.4	1.869	4.486
.5	4.000	2.000	2.5	1.842	4.605
.6	3.582	2.149	2.6	1.817	4.724
.7	3.279	2.295	2.7	1.793	4.841
.8	3.852	2.442	2.8	1.772	4.962
.9	2.865	2.579	2.9	1.751	5.077

1.0	2.718	2.718	3.0	1.732	5.196
1.1	2.593	2.852	3.5	1.650	5.775
1.2	2.488	2.985	4.0	1.587	6.349
1.3	2.398	3.117	4.5	1.537	6.916
1.4	2.319	3.246	5.0	1.495	7.475
1.5	2.250	3.375	5.5	1.460	8.030
1.6	2.189	3.502	6.0	1.431	8.586
1.7	2.134	3.627	6.5	1.405	9.132
1.8	2.085	3.753	7.0	1.383	9.681
1.9	2.040	3.876	8.0	1.346	10.768
2.0	2.000	4.000	9.0	1.316	11.844

This table will be found of continual value in plotting the companion curves under consideration in the remainder of the paper.

8. THE CURVE $y=x^{1/(x-1)}$. Some investigation of the nature of this locus will serve to aid in our further study. Differentiating, we have

$$\frac{dy}{dx}=x^{1/(x-1)}.\frac{x-1-x\log x}{x(x-1)^2}. \tag{11}$$

For negative values of x the tangent is imaginary; hence, there are no contiguous points in the locus for x negative. But if x and y are both positive it is clear that there is a continuous locus.

In quadrant IV, x is positive and y is negative; hence, for all real points, there is involved an even root of a positive quantity. The points thus determined are real for real values of x. Let $x=p/q$, where the fraction is in its lowest terms, p and q having any positive integer values whatever. This is a general value for x real and positive. Then we have

$$y=\left(\frac{p}{q}\right)^{q/(p-q)}.$$

If y has a negative real value, $p-q$ must be even. Hence, both p and q are odd. Now, consider these values of x,

$$\frac{p}{q},\quad \frac{(2n+1)p+1}{(2n+1)q},\quad \frac{(2n+1)p+2}{(2n+1)q},$$

where p and q are both odd and n is a positive integer. For the first and last values there are corresponding real values for y negative; but for the intermediate value y real is positive. If n should be increased at pleasure, remaining always a positive integer, the second and third values of x can be brought indefinitely near the first. But between the first and third values of x, answering to points in the fourth quadrant, there is the intermediate second value above answering to no real point except one in the first quadrant. Therefore, for that part

of the locus which is in the fourth quadrant, x is infinitely discontinuous; that is, the locus has an infinite number of isolated points infinitely near each to another but no two contiguous.

We may in the same manner show that there is an infinitely discontinuous locus in both the second and third quadrant. But it is interesting to note that the discontinuity in the above case arises in a different way from that in these quadrants. In the latter two cases y becomes *imaginary* for the intermediate value of x; but here, besides its imaginary values, it has a real positive value answering to a point in another quadrant.

Confining ourselves now to the real locus we may state the proposition that

The locus of $y=x^{1/(x-1)}$ has a continuous real branch in the first quadrant, and an infinitely discontinuous real branch in each of the other quadrants.

9. Companions in Space to Plane Curves. If we put z for v in the v-expressions and write the general equation of the companion to the plane curve in the form

$$A \qquad y=z^{1/(z-1)},$$
$$B \qquad fx=z^{z/(z-1)},$$

and look upon A and B as the equations of a curve in space, we may call this the companion in space to the curve $y=fx$. Both equations, as before, may be deduced from (1).

10. Companion Surfaces. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be the equation of the parent surface, and let the companion be derived from

$$C^{1/C}=D^{1/D}, \text{ where } C\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2},\ D\equiv 1-\frac{z^2}{c^2}.$$

The companion alone is

$$\frac{z^2}{c^2}=1-v^{1/(v-1)} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=v^{v/(v-1)}.$$

The only values of v which give a continuous locus are positive values, and the only value of $v^{1/(v-1)}$ which can give a real value for z is $v^{1/(v-1)}=1$; for otherwise z is imaginary. For this value, $v^{1/(v-1)}=1$, $z=0$, c being finite, and the entire continuous locus of the companion is the infinite ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\infty,$$

lying in the plane $z=0$.

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ has, for one of its companions, the surface

given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=v^{v/(v-1)} \text{ and } \frac{z^2}{c^2}=v^{1/(v-1)}-1.$$

This surface is symmetrical with reference to the plane $z=0$, and incloses that plane by an infinite elliptic boundary. As v increases from 0 to ∞ the sections parallel to $z=0$ decrease continually, remaining ellipses always, until they reach their limiting size, which is that of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Another companion to the same curve is

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=v^{v/(v-1)} \text{ and } \frac{y^2}{b^2}=1-v^{1/(v-1)}.$$

y has a real value only when $v^{1/(v-1)}=1$. Hence the companion degenerates to $y=0$ and $\frac{x^2}{a^2}-\frac{z^2}{c^2}=\infty$. It is therefore an hyperbola infinitely removed from the origin and lying in the plane $y=0$.

The companion to the sphere $x^2+y^2+z^2=a^2$ is

$$x^2+y^2=v^{v/(v-1)} \text{ and } z^2=a^2-v^{1/(v-1)}.$$

When $a^2=1$, the companion is an infinite circle. When $a^2<1$, the companion is imaginary. When $a^2>1$, the companion is a real surface. Every section by planes parallel to $z=0$ is a circle. For all values of v which make $v^{1/(v-1)}>a^2$ the planes which make circular sections are imaginary.

NOTE ON THE ADDITION THEOREM IN TRIGONOMETRY.

By DR. G. A. MILLER.

When $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$ are substituted for x_1, x_2, y_1, y_2 respectively, in the well known identity

$$(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_1y_2+x_2y_1)^2+(x_1x_2-y_1y_2)^2,$$

there results

$$(\cos^2\alpha+\sin^2\alpha)(\cos^2\beta+\sin^2\beta)=(\cos\alpha\sin\beta+\cos\beta\sin\alpha)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2$$

or

$$(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2=1.$$

For every value of α and β the two expressions in parenthesis represent real

numbers whose squares are together equal to unity. Hence one represents $\sin\phi$ and the other $\cos\phi$, where ϕ is a function of α and β.

Suppose we know that

$$\sin\alpha\cos\beta+\cos\alpha\sin\beta=\sin(\alpha+\beta)=\sin(180^\circ-\alpha-\beta) \qquad (A).$$

It follows from the given identity that

$$\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos(\alpha+\beta) \text{ or } \cos(180^\circ-\alpha-\beta).$$

The latter is impossible since $\alpha=\beta=0$ would not satisfy the equation. Hence we have that from the given identity and (A) it follows that

$$\cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos(\alpha+\beta)=\cos(-\alpha-\beta).$$

If the given identity is written in the form

$$(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_2y_1-x_1y_2)^2+(x_1x_2+y_1y_2)^2$$

we have, by the method used above, that

$$\sin\alpha\cos\beta-\cos\alpha\sin\beta \text{ and } \cos\alpha\cos\beta+\sin\alpha\sin\beta$$

are respectively the sign and cosine of a given angle. It should be observed that it does not matter which of these expressions is taken for sine ϕ except that the formulas come out in a little briefer form if we proceed as above.

This note does not aim at completeness. Its object is to call the attention of teachers of trigonometry to the fact that the derivation of these formulas could proceed along lines which differ from those which are generally pursued, and that these methods offer many advantages. Other things being equal it is always desirable to exhibit connection between different formulas rather than to derive each independently.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

229. Proposed by BENJAMIN FRANKLIN YANNEY, Mount Union College, Alliance, Ohio.

If $a_1^n+a_2^n+a_3^n+\ldots+a_r^n=A^n$, $a_1^m+a_2^m+a_3^m+\ldots+a_r^m>$ or $<A^m$, according as $m<$ or $>n$; provided all the letters stand for positive real numbers.

Solution by the PROPOSER.

We have given $a_1^n+a_2^n+a_3^n+\ldots+a_r^n=A^n\ldots(1)$.

Dividing by A^n, $\left(\frac{a_1}{A}\right)^n+\left(\frac{a_2}{A}\right)^n+\left(\frac{a_3}{A}\right)^n+\ldots\left(\frac{a_r}{A}\right)^n=1\ldots(2)$.

Since the sum of the terms in the left member of (2) is 1, each term is <1. Hence each of the fractions

$$\frac{a_1}{A},\quad \frac{a_2}{A},\quad \frac{a_3}{A},\quad \ldots,\quad \frac{a_r}{A}$$

is a proper fraction. Then, if in (2) we substitute m for n, we shall have

$$\left(\frac{a_1}{A}\right)^m+\left(\frac{a_2}{A}\right)^m+\left(\frac{a_3}{A}\right)^m+\ldots+\left(\frac{a_r}{A}\right)^m > \text{ or } <1,$$

according as $m<$ or $>n$, as is clearly evident. Multiplying by A^m, $a_1^m+a_2^m+a_3^m+\ldots+a_r^m >$ or $<A^m$, according as $m<$ or $>n$.

271. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1)\ldots(y-x)+2y(2y-1)\ldots(2y-x)+\ldots+zy(zy-1)\ldots(zy-x)$.

Dr. Zerr obtains $\frac{1}{x!}\sum_{r=y}^{r=zy}\int_0^1\left[\log\left(\frac{1}{u}\right)\right]^r du$ as the sum of the series. This does not satisfy the requirements of the problem for the reason that the sum is to be in integral form. ED. F.

272. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the relations $x=\frac{ar+bs}{\lambda}=\frac{as-br}{\mu}=\frac{a\lambda-b\mu}{r}=\frac{a\mu+b\lambda}{s}$ between the finite real quantities x, a, b, r, s, λ, μ requires that $x^2=a^2+b^2$.

I. Solution by the PROPOSER.

These relations make the determinant

$$\Delta\equiv\begin{vmatrix} \lambda x+i\mu x & (ar+bs)+i(as-br) \\ (a\lambda-b\mu)+i(a\mu+b\lambda) & rx+isx \end{vmatrix},\quad (i=\sqrt{-1}),$$

necessarily $=0$; for its columns are identical. Dividing the first column by $\lambda+i\mu$ and the second by $r+is$, we have

$$\Delta\equiv(\lambda+i\mu)(r+is)\begin{vmatrix} x & a-ib \\ a+ib & x \end{vmatrix}=0.$$

Hence,

$$\begin{vmatrix} x & a-ib \\ a+ib & x \end{vmatrix}=0;\ \text{or } x^2=a^2+b^2.$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and A. H. HOLMES, Brunswick, Me.

By clearing of fractions, adding the four resulting equations and solving for $x-a$, we have

$$x-a=b(\lambda-\mu-\nu+s)/(\lambda+\mu+\nu+s)\ldots(1).$$

Also, after the equations are cleared of fractions, by subtracting the sum of the second and third from the sum of the first and fourth, and solving for $x+a$, we have

$$x+a=b(\lambda+\mu+\nu+s)/(\lambda-\nu-\mu+s)\ldots(2).$$

Multiplying (1) by (2), we have $x^2-a^2=b^2$ or $x^2=a^2+b^2$.

Also solved by G. W. Greenwood and G. B. M. Zerr.

Professor Greenwood furnished two solutions, the first of which made use of the vanishing of the determinant of the coefficients of λ, μ, ν, s in order that the four equations be consistent. The second solution was obtained by finding the values of μ/ν, ν/λ, s/λ in any three of the four equations and substituting them in the remaining equation.

Dr. Zerr solved each of the four equations for $x\lambda$, $x\mu$, $x\nu$, and xs, respectively, and added the square of the resulting equations. By dividing this last equation by $\Sigma\lambda^2$, x^2 is immediately found.

Professors Zerr and Greenwood also sent in solutions of 271.

GEOMETRY.

300. Proposed by J. J. QUINN, Ph. D., Scottdale, Pa.

Trisect an angle by means of a tractrix.

No solution has been received.

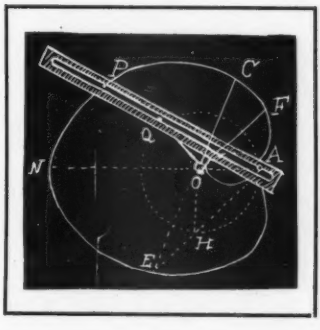
301. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $r=a(1+2\cos\theta)$ to the trisection of an angle. Describe the curve by continuous motion.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

To describe the curve by continuous motion, we may construct an instrument of the following description:

Take a bar AP and cut a path, indicated in the figure by the continuous white line, equal in length to four times the radius OA of the circle. Fasten the radius OQ at the middle point, Q, of the bar AP by means of a joint. Fasten the other end, O, of the radius at O. At P, a distance from Q equal to OQ, insert a pen point. Then if the path in the bar is set over a fixed pin at A', at a distance from O equal to OQ, the point P will describe the curve as the bar is made to move so as to keep A in the path of the bar.



To trisect an angle, for example, the angle AOC, by means of this curve, produce CO to E and draw EA. Also draw OH; then FO drawn parallel to EA makes the angle $FOC=\frac{1}{3}\angle AOC$. For since $EH=HO$, by construction of the curve, $\angle OEH=\angle EOH$. But $\angle OHA,=\angle OAH, =\angle OEH+\angle EOH=2\angle OEH$. Hence, $\angle OEH+\angle OAE, =3\angle OEH, =\angle AOC$, or $\angle OEH, =\angle FOC=\frac{1}{3}\angle AOC$.

After this department, in the last issue, was in type, we received solutions of problem 299 from Professors Scheffer, Zerr, and Greenwood. Professors Scheffer and Greenwood's solutions consisted in connecting a point, G, of the ellipse with the foci F, F'. M, the middle point of FG, is taken for the center of the circle described on the focal radius, FG, as a diameter. The line AM joining M and A, the center of the ellipse, is $\frac{1}{2}F'G$, since $AF=AF'$ and M is the middle point of FG. But $\frac{1}{2}AF'=\frac{1}{2}(2a-AF)=a-\frac{1}{2}AF$, from the definition of the ellipse. Hence, MA, the distance between the centers of the auxiliary circle and the circle described on $AF=a-\frac{1}{2}AF$, the difference of their radii. Hence the circles touch.

Dr. Zerr's solution, which was analytical, made use of the same property.

CALCULUS.

228. Proposed by B. F. FINKEL, Ph. D., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A sphere, radius r, is dropped into a conical vessel whose vertex angle is 60°. Find the contents of the vessel between the vertex and the sphere by means of the formula, $V=\int\int\int dx\,dy\,dz$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and the PROPOSER.

$x^2+y^2+z^2=r^2$ is the equation to the sphere, and $x^2+y^2=\frac{1}{3}(2r-z)^2$ is the equation to the cone. Eliminating z we get $y=\sqrt{(\frac{3}{4}r^2-x^2)}$.

$\therefore y=\sqrt{(\frac{3}{4}r^2-x^2)}=y'$ to $y=0$, $x=\frac{1}{2}r\sqrt{3}=x'$ to $x=0$.

$$\therefore v=4\int_0^{x'}\int_0^{y'}[2r-\sqrt{3}\sqrt{(x^2+y^2)}-\sqrt{(r^2-x^2-y^2)}]dxdy$$

$$=4\int_0^{x'}\left[r\sqrt{(\tfrac{3}{4}r^2-x^2)}-\tfrac{1}{2}(r^2-x^2)\sin^{-1}\sqrt{\frac{\frac{3}{4}r^2-x^2}{r^2-x^2}}\right.$$

$$\left.-\tfrac{1}{2}\sqrt{3}\,x^2\log\left(\frac{\sqrt{(\frac{3}{4}r^2-x^2)}+\frac{1}{2}r\sqrt{3}}{x}\right)\right]dx$$

$$=4(\tfrac{3}{16}\pi r^3-\tfrac{1}{64}\pi r^3+\tfrac{1}{12}\pi r^3-\tfrac{1}{6}\pi r^3-\tfrac{3}{64}\pi r^3)=\tfrac{1}{6}\pi r^3.$$

229. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Solve the differential equation $d^2y/dx^2=axy$.

Solution by S. A. COREY, Hiteman, Iowa, and LEROY D. WELD, Coe College, Cedar Rapids, Iowa.

Let $y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+$ etc. ...(1).

Then $d^2y/dx^2=2c_2+6c_3x+12c_4x^2+20c_5x^3+30c_6x^4+$ etc. ...(2),
and $axy=ac_0x+ac_1x^2+ac_2x^3+ac_3x^4+ac_4x^5+$ etc. ...(3).

Equating coefficients of like powers of x in (2) and (3), and reducing, we obtain by substituting in (1),

$$y=c_0\left(1+\frac{ax^3}{3!}+\frac{4a^2x^6}{6!}+\frac{4.7a^3x^9}{9!}+\frac{4.7.10a^4x^{12}}{12!}+\text{etc.}\right)$$

$$+c_1\left(x+\frac{2ax^4}{4!}+\frac{2.5a^2x^7}{7!}+\frac{2.5.8a^3x^{10}}{10!}+\frac{2.5.8.11a^4x^{13}}{13!}+\text{etc.}\right)$$

Also solved similarly by G. B. M. Zerr. Professor William Hoover did not give a solution but referred to the discussion of the general problem in Forsythe's *Differential Equations*, Chapter VII, page 217, Second Edition. A discussion of the solution of this class of differential equations, by definite integrals, is also given in Price's *Infinitesimal Calculus*, Vol. II., page 484.

MECHANICS.

187. Proposed by M. E. GRABER, M. A., Heidelberg University, Tiffin, Ohio.

Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The equations of motion are $d^2x/dt^2+\mu x=0$ and $d^2y/dt^2+\mu y=0$.

$\therefore x=A\cos t\sqrt{\mu}+B\sin t\sqrt{\mu}$, and $y=C\cos t\sqrt{\mu}+D\sin t\sqrt{\mu}$.

$\therefore (Ay-Cx)^2+(By-Dx)^2=(AD-BC)^2$, an ellipse with center of force at center.

192 Proposed by WILLIAM HOOVER, Ph D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A solid sphere rolls down a trough formed by two planes which make with each other an angle 2α. Find, by the principle of *vis viva*, the expression for the time of rolling down the trough when the inclination of the trough to the horizon is β.

Solution by LEROY D. WELD, Coe College, Cedar Rapids, Iowa, and the PROPOSER.

Let C be the center of the sphere, O the center of the line joining the points of contact A and B of the trough and sphere, k the radius of gyration of the sphere about its center, r=its radius, θ=the angle through which a fixed radius in the plane of O and the edge of the angle 2ϕ, has rotated in any time t from the beginning of motion; let a plane and line, the first embracing AB, and the other passing through C, be drawn parallel to the edge of 2ϕ, both cutting a fixed horizontal plane, the line in the point D; x, y, the coördinates of C at the time t, D, the origin, and a and b the initial values of x, y; then by *vis viva*, m being the mass of the sphere,

$$m\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}\right)=C-2mgy.$$

When $y=b$, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=0$, $\frac{d\theta}{dt}=0$; $\therefore C=2mgb$, and we have

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}=2g(b-y)\ldots(1).$$

Now $x=a-\theta r\sin\alpha\cos\beta$, $y=b-\theta r\sin\alpha\sin\beta$; then

$$\frac{dx}{dt}=-r\sin\alpha\cos\beta\frac{d\theta}{dt},\quad \frac{dy}{dt}=-r\sin\alpha\sin\beta\frac{d\theta}{dt},\quad \frac{dx^2+dy^2}{dt^2}=r^2\sin^2\alpha\frac{d\theta^2}{dt^2};\text{ then}$$

$$k^2\frac{d\theta^2}{dt^2}=\frac{k^2}{r^2\sin^2\alpha}\cdot\frac{dx^2+dy^2}{dt^2}.$$

Substituting in (1),

$$\frac{dx^2+dy^2}{dt^2}\cdot\frac{r^2\sin^2\alpha+k^2}{r^2\sin^2\alpha}=2g(b-y)=2g\theta r\sin\alpha\sin\beta\ldots(2).$$

But $dx^2+dy^2=ds^2$, in which $s=\theta r\sin\alpha$, or $ds=r\sin\alpha d\theta$; $\therefore$ (2) becomes

$$\frac{ds^2}{dt^2}=\frac{2r^2\sin^2\alpha}{r^2\sin^2\alpha+k^2}gs\sin\beta\ldots(3).$$

Taking the derivative of both members of (3) with respect to t, dividing by $\frac{ds}{dt}$, multiplying by dt and integrating twice, noticing that when $t=0$, $\frac{ds}{dt}=0$, and $s=0$, and finally putting $s=l=$the length of the trough,

$$s=\tfrac{1}{2}\frac{r^2\sin^2\alpha}{r^2\sin^2\alpha+k^2}gt^2,\text{ whence }t=\frac{1}{\sin\alpha}\sqrt{\frac{(10\sin^2\alpha+4)l}{5g\sin\beta}},\ k^2\text{ being }\tfrac{2}{5}r^2.$$

Also solved by G. B. M. Zerr and G. W. Greenwood.

DIOPHANTINE ANALYSIS.

136 Proposed by A H HOLMES, Brunswick, Maine.

In the equation in Diophantine Analysis: $2x^2+2x+1=\square=u^2$, show that u is always the sum of two squares.

Solution by L. E. NEWCOMB, Los Gatos, Cal.

$2x^2+2x+1=x^2+(x+1)^2\ldots(1)$.

Let $pq=x$ or $pq=x+1$ according as x is odd or even; then, for all integral values of x that satisfy (1), $\frac{1}{2}p^2-\frac{1}{2}q^2=x+1$ or $\frac{1}{2}p^2-\frac{1}{2}q^2=x$.

$\therefore p^2q^2+(\frac{1}{2}p^2-\frac{1}{2}q^2)^2=x^2+(x+1)^2=u^2$.

But $p^2q^2+(\frac{1}{2}p^2-\frac{1}{2}q^2)^2=(\frac{1}{2}p^2+\frac{1}{2}q^2)$. For p, substitute $m+n$, for q, $m-n$; then $(\frac{1}{2}p^2+\frac{1}{2}q^2)^2$ becomes $(m^2+n^2)^2$. Since $(m^2+n^2)^2=u^2$, $u=m^2+n^2$), the sum of two squares.

AVERAGE AND PROBABILITY.

172. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A circular arc, with center at one corner of a given square, is drawn through a point taken at random in the square. What is the average length of the arc within the square?

II. Solution by HENRY HEATON, Belfield, N. D.

Let r=distance of the random point from the corner and n=number of points in the unit area. If $r<a$, the length of the arc is $\frac{1}{2}\pi r$, and the number of points from r to $r+dr$ is $\frac{1}{4}n\pi r dr$. Hence, the number of arcs of length $\frac{1}{2}\pi r$ is $\frac{1}{4}\pi n r dr$. If $r>a$ and $<a\sqrt{2}$ the length of the arc is $r[\frac{1}{2}\pi-2\cos^{-1}(a/r)]$ and the number of arcs of this length is $nr[\frac{1}{2}\pi-2\cos^{-1}(a/r)]$. To find the average, we must multiply the length of each arc by the number of times it is drawn, take the sum of the products, and divide the result by the whole number of arcs.

Hence, the required average is

$$A=\{n\int_0^a \tfrac{1}{4}\pi^2 r^2 dr+n\int_a^{a\sqrt{2}} r^2[\tfrac{1}{2}\pi-2\cos^{-1}(a/r)]^2\}\div na^2$$

$$=\frac{\pi^2}{4a^2}\int_0^{a\sqrt{2}} r^2 dr+\frac{1}{a^2}\int_a^{a\sqrt{2}} 4\{[\cos^{-1}(a/r)]^2-2\pi\cos^{-1}(a/r)\}r^2 dr$$

$$=\tfrac{2}{3}a\int_0^{\frac{1}{4}\pi}(\pi-4\theta)\sec^3\theta d\theta, \text{ where } \theta=\cos^{-1}(a/r)$$

$$=\tfrac{4}{3}a[\sqrt{2}-1+\int_0^{\frac{1}{4}\pi}\log(\tan\theta+\sec\theta)d\theta]=(1.03+)a.$$

NOTE. Since the publication of our solution of this problem in the October number of the MONTHLY, we received a criticism from Mr. Heaton, to the effect that the published solution is wrong. Mr. Sanders also agrees with Mr. Heaton in his contention. Of course we expected that criticism would follow, and do not now hope to forever settle the contention that arises when problems of this sort are proposed in the *indefinite* form. However, we again repeat what we have several times said before, that when no law of distribution is assigned there are as many ways of solving the problem as there are ways of assigning laws of distribution. In such cases, there is no such thing as *the* correct solution. The above problem is stated in the indefinite form, and so one solution assuming one law of distribution is as good as another assuming some other law of distribution. We have published Mr. Heaton's elegant solution for the benefit of those who have interpreted the problem in his way. Ed. F.

177 (Incorrectly numbered 176). Proposed by T. N. HAUN, Mohawk, Tenn.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? (Problem 72, p. 503, Williamson's *Integral Calculus*.)

A solution of this problem by Dr. Neikirk is given on pp. 180-182, Vol. VI, of MONTHLY. If any one has a different solution to offer, we shall be glad to consider it.

178. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

No solution has been received.

179. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and LEROY D. WELD, Coe College, Cedar Rapids, Ia.

If the chord varies as the abscissa we get, since $x^2+y^2=2ax$ is the equation to the circle,

$$M=\frac{\int_0^{2a} y(2ax)dx}{\int_0^{2a} dx}=\tfrac{4}{3}a.$$

If the chord varies as the angle it makes with the diameter through its extremity, then, since $r=2a\cos\theta$ is the equation to the circle, we get

$$M=\frac{\int_0^{\frac{1}{2}\pi} 2a\cos\theta\, d\theta}{\int_0^{\frac{1}{2}\pi} d\theta}=\frac{4a}{\pi}.$$

MISCELLANEOUS.

161 (Incorrectly numbered 157). Proposed by H. L. ORCHARD, M. A., B. Sc. (Unsolved problem in the Educational Times, London.)

An inelastic rod 9 feet long is placed with its upper end upon a rough vertical plane and its lower end on a smooth horizontal plane, and so that it makes an angle of 45° with each plane. It is now released, and strikes against a smooth sphere of 1 foot diameter placed in contact with the two planes. Determine the subsequent motion.

No solution has been received.

163. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two straight streams of different volumes and velocities come together. Find the path of a body floating in mid-current of either.

No solution has been received.

PROBLEMS FOR SOLUTION.

ALGEBRA.

278. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$xyz((\Sigma x)^2 < 3\Sigma y^2 z\ \Sigma yz^2$, if x, y, z are positive.

GEOMETRY.

309. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

To find the equation of Brocard's Ellipse, the sides b and c of the triangle being the axes of coördinates.

310. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Construct a plane triangle having given the base, the vertical angle, and the bisector of the vertical angle.

CALCULUS.

234. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Find the first negative pedal of an ellipse semi-axes a, b referred to origin as center, and show that its entire area is $\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right]$.

MECHANICS.

198. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Three spheres of the same material, radii R, r, S, rest upon a horizontal plane, touching each other. Find the radius of a sphere of the same material as the others which, being placed upon the other three spheres, will just prevent the latter from separating, the coefficient of friction between the spheres being μ, and between the spheres and the table being μ'.

DIOPHANTINE ANALYSIS.

145. Proposed by JOHN D. WILLIAMS (being the twelfth of his 14 challenge problems proposed in 1832).

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\frac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

AVERAGE AND PROBABILITY.

165. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If a line l is divided into n parts by $n-1$ points taken at random on it, what is the mean value of the pth power of one of the parts taken at random?

MISCELLANEOUS.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Sum to n terms, $\sin\alpha \sin\beta + \sin\alpha - \beta \sin\beta + \gamma + \sin\alpha - 2\beta \sin\beta + 2\gamma + \ldots$

NOTES AND NEWS.

Mr. George Gailey Chambers has been appointed an instructor in mathematics at the University of Pennsylvania.

J. N. Ambler, Professor of Mathematics in Roanoke College, Salem, Va., has resigned to accept a position as city engineer of Winston, N. C.

Mr. Chester E. Dimick has resigned an instructorship at the University of Pennsylvania to become an instructor in mathematics on the U. S. S. Chase, of the revenue service.

We regret to announce to our readers that, owing to increased duties at The University of Chicago and to his desire to devote all his leisure moments to investigations in a most attractive field of research in which he is now interested, Editor Dickson has decided to withdraw from the active editorship of the MONTHLY. However, he has agreed to remain with us in a semi-official way, rendering us such assistance from time to time as his leisure will permit. The work Dr. Dickson has been doing so efficiently for the past four years will now be done by Dr. H. E. Slaught, Assistant Professor of Mathematics in the University of Chicago, and Secretary of the Chicago Section of the American Mathematical Society. While we are very sorry to lose the active service of Dr. Dickson, we are glad to have his work taken up by so capable a man as Dr. Slaught. All articles for publication in the MONTHLY should in the future be sent to him instead of to Dr. Dickson.

We take this opportunity in behalf of the readers of the MONTHLY to extend to Dr. Dickson our sincere thanks for the valuable service he has rendered, through the MONTHLY, to the science of mathematics. F.

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ARITHMETIC. Fisk's *Foundations of Higher Arithmetic*, p. 40; *Wentworth's New Elementary Arithmetic*, p. 114.

CALCULUS. *Chandler's Elements of the Infinitesimal Calculus*, p. 113; Johnson's *Treatise on the Integral Calculus*, p. 211; Osgood's *First Course in the Differential and Integral Calculus*, p. 190.

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